# Global well-posedness of 3D Burgers equation with a multiplicative noise force

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# Outline

Based on joint work with Zhao Dong (Chinese Academy of Sciences, Beijing) and Guoli Zhou (Chongqing Univesity).

# Outline

1. Introduction to 3D stochastic Burgers equation

2. Well-posedness and long-term behavior: constant diffusion coefficient
   - Local well-posedness
   - Global well-posedness
   - Long-time behaviour

3. Global well-posedness : function diffusion coefficient

# 3D stochastic Burgers equation

Let $\mathbb{T}^3 = \mathbb{R}^3/2\pi\mathbb{Z}^3$ denote the 3-dimensional torus, we are concerned with the following 3D stochastic Burgers equation

$$\begin{cases} d\mathbf{u}(t,x) - \Delta\mathbf{u}(t,x)dt + (\mathbf{u}\cdot\nabla\mathbf{u})(t,x)dt = \mathbf{u}(t,x)\circ b(x)dB(t), \quad \textit{on } [0,T]\times\mathbb{T}^3, \\ \mathbf{u}(0,x) = \mathbf{u}_0(x), x = (x_1,x_2,x_3)\in\mathbb{T}^3 \end{cases} \tag{1}$$

for an unknown velocity field $\mathbf{u}(t,x) = (u_1(t,x), u_2(t,x), u_3(t,x))$, which is interpreted as a random field,

where

$b(x) : \mathbb{T}^3 \to \mathbb{R}$, $b(x) \in C^\infty(\mathbb{T}^3)$,

$B(t)$ is a standard one dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$.

# Background and known results

Background:

- The deterministic model was first introduced by H.Bateman in 1915 and studied mathematically by J.M.Burgers in 1939. This model is used to describe both nonlinear propagation effects and diffusive effects, occurring in various areas of applied mathematics, such as gas dynamics, fluid mechanics, nonlinear acoustics, and (more recently) traffic flows.

Known results (via maximum principle):

- Kiselev and Ladyzhenskaya (1957) proved global well-posedness for 3D deterministic Burgers equation in $L^\infty([0,T]; L^\infty(\mathcal{O})) \cap L^2([0,T]; H_0^1(\mathcal{O}))$ .
- When the viscosity tends to zero and the initial condition is zero, Bui (1975) showed the convergence of solutions of the deterministic 3D Burgers equation to the inviscid Burgers equation (local solution).
- Robinson, Rodrigo, and Sadowski (2016) established the global well-posedness of classical solutions of 3D deterministic Burgers equation .
- Brzezniak, Goldys, Neklyudov (2014) studied the global existence and uniqueness of mild solutions in $L^p(\mathbb{T}^3)$ and $L^p(\mathbb{R}^3)$, $p > 3$, for the 3D stochastic Burgers equation with additive noise.

# Known results

For potential multidimensional Burgers equation, though the Hopf-Cole transformation, one can reduce the equation either to the heat equation or to the Hamilton–Jacobi equation, so that the corresponding gradient solution or the viscosity solution are studied, see e.g., R. Iturriaga and K. Khanin (2003), D. Gomes, R.Iturriaga, K. Khanin, and P. Padilla (2005), Y. Bakhtin, E. Cator and K. Khanin (2014), A.Boritchev (2016), Y. Bakhtin and L. Li (2019), A. Dunlap, C. Graham and L. Ryzhik (2022), just mention a few.

Our aim: For the no-potential stochastic 3D Burgers equation (1), utilising the so-called Doss-Sussman transformation, we want to study regularities and long-time behaviour of its solutions.

# Notations and setup

- For $1 \leq p \leq \infty$, $\mathbb{L}^p(\mathbb{T}^3)$ denotes the Lebesgue spaces $\mathbb{L}^p(\mathbb{T}^3; \mathbb{R}^3)$ with the norm $|\cdot|_p$.When $p = 2$, $\langle \cdot, \cdot \rangle$ represents the inner product in $\mathbb{L}^2(\mathbb{T}^3)$. For $s \geq 0$, we introduce an operator $\Lambda^s$ acting on the Sobolev space $\mathbb{H}^s(\mathbb{T}^3) := \mathbb{H}^s(\mathbb{T}^3; \mathbb{R}^3)$.
- Let $f \in \mathbb{H}^s(\mathbb{T}^3)$ with the Fourier series

$$f(x) = \sum_{k \in \mathbb{Z}^3} \hat{f}_k e^{ik \cdot x} \in \mathbb{H}^s(\mathbb{T}^3),$$

- we define

$$\Lambda^s f(x) = \sum_{k \in \mathbb{Z}^3} |k|^s \hat{f}_k e^{ik \cdot x} \in \mathbb{L}^2(\mathbb{T}^3).$$

Obviously, $\Lambda^2 = -\Delta$. Denote by $\| \cdot \|_s$ the seminorm $|\Lambda^s \cdot|_2$, then the Sobolev norm $\| \cdot \|_{\mathbb{H}^s}$ of $\mathbb{H}^s(\mathbb{T}^3)$ is equivalent to $| \cdot |_2 + \| \cdot \|_s$.

- Denote by $\| \cdot \|_s$ the seminorm $|\Lambda^s \cdot|_2$, then the norm $\| \cdot \|_{\mathbb{H}^s}$ of $\mathbb{H}^s(\mathbb{T}^3)$ is equivalent to $| \cdot |_2 + \| \cdot \|_s$ and we can then define the norm

$$\|f\|_{\mathbb{H}^s} := \Big( \sum_{k \in \mathbb{Z}^3} (1 + |k|^{2s}) |\hat{f}_k|^2 \Big)^{1/2}.$$

# The Doss-Sussman transformation

Assume that $b(x) = b$ (a constant). Let $\alpha(t) := \exp\left(-bB(t)\right), t \in [0, T]$. Taking the Doss-Sussman transformation $\mathbf{v} = \alpha \mathbf{u}$ then yields the following equivalent form of (1),

$$\begin{cases} d\mathbf{v}(t,x) - \Delta \mathbf{v}(t,x)dt + \alpha^{-1}(t)(\mathbf{v} \cdot \nabla \mathbf{v})(t,x)dt = 0, & on\ [0, T] \times \mathbb{T}^3, \\ \mathbf{v}(0,x) = \mathbf{u}_0(x),\ x = (x_1, x_2, x_3) \in \mathbb{T}^3. \end{cases} \tag{2}$$

Notice that Equation (2) is a nonlinear parabolic PDE with a random coefficient.

Comparing deterministic and stochastic 3D Burgers equations

1. If $b = 0$ (the deterministic 3D Burgers equation), the maximum principle can be applied to the classical local solution to deduces its global existence.
   If $b \neq 0$ (the stochastic 3D Burgers equation), our idea is to apply the maximum principle to the random Galerkin approximation and establish some *a priori* estimates.

2. If $b = 0$, there is no long-time behaviour result.
   If $b \neq 0$ ergodicity can be established.

# Definitions of solutions to Equation (2)

## Definition (Local strong/weak solutions to Equation (2))

Let $T \in (0, \infty)$ be arbitrarily fixed. Suppose $\mathbf{u}_0 \in \mathbb{H}^1(\mathbb{T}^3)/\mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)$ is an $\mathcal{F}_0$-measurable random variable.

1. A pair $(\mathbf{v}, \tau)$ is a local strong pathwise solution to (2) if $\tau$ is a strictly positive random variable taking values in $(0, \infty)$ and $\mathbf{v}(\cdot \wedge \tau)$ satisfies (2) in a weak sense with the following regularities (note: statements hold almost surely),

$$\mathbf{v}(\cdot \wedge \tau) \in C([0, T]; \mathbb{H}^1(\mathbb{T}^3)) \cap L^2([0, T]; \mathbb{H}^2(\mathbb{T}^3))/C([0, T]; \mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)) \cap L^2([0, T]; \mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3)),$$

   and

$$\partial_t \mathbf{v}(\cdot \wedge \tau) \in L^1([0, T]; \mathbb{L}^2(\mathbb{T}^3)).$$

2. Strong/weak pathwise solutions of (2) are said to be pathwise unique up to a random positive time $\tau > 0$ if given any pair of solutions $(\mathbf{v}^1, \tau), (\mathbf{v}^2, \tau)$ which coincide at $t = 0$ on the event $\tilde{\Omega} = \{v^1(0) = v^2(0)\} \subset \Omega$, then

$$\mathbb{P}(I_{\tilde{\Omega}}(\mathbf{v}^1(t \wedge \tau) - \mathbf{v}^2(t \wedge \tau)) = 0; \forall t \in [0, T]) = 1.$$

# Definitions of solutions to Equation (2)

## Definition (Maximal and global strong/weak solutions to Equation (2))

(i) Let $\xi$ be a positive random variable which may take $\infty$ at some $\omega \in \Omega$. We say the pair $(\mathbf{v}, \xi)$ is a maximal pathwise strong/weak solution if for any random variable $\tau \in (0, \xi)$, $(v, \tau)$ is a local strong/weak pathwise solution satisfying

$$\sup_{t\in[0,\tau]} \|\mathbf{v}(t)\|_1 < \infty, \quad \text{and} \quad \limsup_{t\to\xi} I_{[\xi<\infty]} \|\mathbf{v}(t)\|_1 = \infty, \quad a.s..$$

(ii) If $(\mathbf{v}, \xi)$ is a maximum pathwise strong/weak solution and $\xi = \infty$ a.s., then we say the solution $\mathbf{v}$ is global.

# Outline

# Galerkin approximation of Equation (2)

For $n \in \mathbb{N}$, let $P_n$ denote the projection on to the Fourier modes of order up to $n$, set

$$\mathbf{v}_n := P_n \left( \sum_{k \in \mathbb{Z}^3} \hat{v}_k e^{ix \cdot k} \right) = \sum_{|k| \leq n} \hat{v}_k e^{ix \cdot k}.$$

The Galerkin approximation of (2) is then given for each $n \in \mathbb{N}$ by

$$\begin{aligned} d\mathbf{v}_n(t,x) &= \Delta \mathbf{v}_n(t,x)dt - \alpha^{-1}(t) P_n[(\mathbf{v}_n \cdot \nabla \mathbf{v}_n)(t,x)]dt, \quad \text{on } [0,T] \times \mathbb{T}^3, \\ \mathbf{v}_n(0,x) &= \mathbf{u}_n(0,x) = P_n \mathbf{u}(0,x), \quad x = (x_1, x_2, x_3) \in \mathbb{T}^3. \end{aligned} \tag{3}$$

Since (3) defines a locally-Lipschitz system of random ODEs, it is clear that for each $n \in \mathbb{N}$ there is a unique local solution $\mathbf{v}_n$ associated with initial $\mathbf{v}_n(0,x) \in \mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)$. Define

$$\tau_n = \inf\{t \in \mathbb{R}^+ : \sup_{0 \leq s \leq t} \|\mathbf{v}_n(s)\|_{\mathbb{H}^{\frac{1}{2}}} = \infty\}.$$

Obviously, $\mathbf{v}_n \in C([0,\tau_n) \times \mathbb{T}^3)$.

## Poincaré inequality for Galerkin approximation (3)

Let $\mathbf{u}, \mathbf{v}$ be the corresponding local solutions of (3) up to a random positive time $\tau > 0$ with initial data $\mathbf{u}_0 \in \mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)$,$\mathbf{v}_0 \in \mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)$ respectively. Let $\xi := \mathbf{u} - \mathbf{v}$ and $\xi_0 := \mathbf{u}_0 - \mathbf{v}_0$, then for $t \in [0, \tau]$, we have

1. $$\left|\int_{\mathbb{T}^3} \Big(\xi(t) - \xi_0\Big) dx\right| \le 8\pi^3 \int_0^t \alpha^{-1}(s)\|\xi\|_{\frac{1}{2}}(\|\mathbf{u}(s)\|_{\frac{1}{2}} + \|\mathbf{v}(s)\|_{\frac{1}{2}})ds.$$

2. In particular, taking $v \equiv 0$ yields the following
$$\left|\int_{\mathbb{T}^3} \mathbf{u}(x,t)dx\right| \le 8\pi^3 \int_0^t \alpha^{-1}(s)\|\mathbf{u}(s)\|^2_{\frac{1}{2}}\, ds + \left|\int_{\mathbb{T}^3} \mathbf{u}_0(x)dx\right|.$$

3. For any $s > 0$ and $t \in [0, \tau]$, we have
$$\|\mathbf{v}_n(t)\|_s \le \|\mathbf{v}_n(t)\|_{\mathbb{H}^s} \le c\|\mathbf{v}_n(t)\|_s + c\int_0^t \|\mathbf{v}_n(s)\|^2_{\frac{1}{2}}\, ds + c|\mathbf{u}_0|_1, c = c(\mathbb{T}^3, \alpha^{-1}) > 0.$$

# Maximum principle for Galerkin approximation Equation(3)

## Lemma 1 (maximum principle for Equation (3))

*If $\mathbf{v}_n$ is a solution to the random Burgers equation (3) on the time interval $[0, \tau_n)$, then*

$$\sup_{s\in[0,\tau_n)} |\mathbf{v}_n(s)|_\infty \quad \leq \quad |\mathbf{v}_n(0)|_\infty, \quad \mathbb{P} - a.s.\omega \in \Omega.$$

Sketch of proof: Let $\beta > 0$ and set $f(s) := f(s,x) := e^{-\beta s}\mathbf{v}_n(s,x)$ for all $s \in [0, \tau_n)$ and $x \in \mathbb{T}^3$. Then, we have

$$\partial_s |f(s)|^2 + 2\beta |f(s)|^2 + e^{\beta s}\alpha^{-1}(s) f(s) \cdot \nabla |f(s)|^2 - \Delta |f(s)|^2 + 2|\nabla f|^2 = 0.$$

We observe that if $|f|$ has local maximum at $(t,x) \in (0,\tau_n) \times \mathbb{T}^3$, then the left hand side of the above equality is positive unless $|f(t,x)| \equiv 0$. Therefore,

$$|f(s)|_\infty \leq |f(0)|_\infty,$$

which implies

$$|\mathbf{v}_n(s)|_\infty \leq e^{\beta s}|\mathbf{v}_n(0)|_\infty, \text{ for } s \in (0, \tau_n).$$

Let $\beta$ tends to $0$, we get the desired result.

# Steps to establish the global well-posedness of 3D stochastic Burgers equation in $\mathbb{H}^1(\mathbb{T}^3)$

Step 1: Applying the maximum principle to Galerkin approximations to establish energy estimates in $\mathbb{H}^1(\mathbb{T}^3)$.

Step 2: For initial data $\mathbf{u}_0 \in \mathbb{H}^1(\mathbb{T}^3)$, find a subsequence of Galerkin approximation $\mathbf{v}_n(s) \to \mathbf{v}(s)$ in $\mathbb{H}^1(\mathbb{T}^3)$, for $s \in [0, \tau(\mathbf{u}_0, \omega))$.

Step 3: For initial data $\mathbf{u}_0 \in \mathbb{H}^1(\mathbb{T}^3)$, and some $t_0 > 0$, find a subsequence of Galerkin approximation $\mathbf{v}_n(s) \to \mathbf{v}(s)$ in $\mathbb{H}^2(\mathbb{T}^3)$, for $s \in [t_0, \tau(\mathbf{u}_0, \omega))$.

Step 4: With the convergence established in Step 2 and Step 3, we prove that the local strong solution will not blowup in any finite time in $\mathbb{H}^1(\mathbb{T}^3)$, which means the global existence of the strong solution.

Here note that the notation $\tau(\mathbf{u}_0, \omega)$ represents the maximum existence time for the local strong solution in $\mathbb{H}^1(\mathbb{T}^3)$.

# Step 1: The maximum principle to Galerkin approximation (3)

## Lemma 2

*For initial data* $\mathbf{v}_n(0,x) \in \mathbb{H}^1(\mathbb{T}^3)$ *and for* $0 < \epsilon < t < \tau_n$*, we have the estimate of* $\mathbf{v}_n$

$$\|\mathbf{v}_n(t)\|_1^2 + \int_\epsilon^t \|\mathbf{v}_n(s)\|_2^2 ds \leq c\|\mathbf{v}_n(\epsilon)\|_1^2 \exp\left(c\|\mathbf{v}_n(\epsilon)\|_{\mathbb{H}^2}^2 \int_0^t \alpha^{-2}(r)dr\right).$$

**Proof.** Taking inner product of (3) with $\Lambda^2\mathbf{v}_n$ in $\mathbb{L}^2(\mathbb{T}^3)$ yields

$$\partial_t\|\mathbf{v}_n\|_1^2 \leq 2\alpha^{-1}\left|\int_{\mathbb{T}^3}(\mathbf{v}_n\cdot\nabla)\mathbf{v}_n\Delta\mathbf{v}_n dx\right| - 2\|\mathbf{v}_n\|_2^2 \leq \alpha^{-2}|\mathbf{v}_n|_\infty^2\|\mathbf{v}_n\|_1^2 + \|\mathbf{v}_n\|_2^2.$$

For $0 < \epsilon < t < \tau^*$, applying maximum principle and some inequalities yields,

$$\|\mathbf{v}_n(t)\|_1^2 + \int_\epsilon^t \|\mathbf{v}_n(s)\|_2^2 ds \leq c\|\mathbf{v}_n(\epsilon)\|_1^2 \exp\left(c\|\mathbf{v}_n(\epsilon)\|_{\mathbb{H}^2}^2 \int_0^t \alpha^{-2}(r)dr\right).$$

# Step 2: Local well-posedness to Equation (2) in $\mathbb{H}^1(\mathbb{T}^3)$: $\mathbf{u}_0 \in \mathbb{H}^1(\mathbb{T}^3)$, $\mathbf{v}_n(s) \to \mathbf{v}(s)$ in $\mathbb{H}^1(\mathbb{T}^3), s \geq 0$.

**Proposition 3 (Local well-posedness of strong solution to Equation (2) in $\mathbb{H}^1(\mathbb{T}^3)$)**

*Suppose $\mathbf{u}_0 \in \mathbb{H}^1(\mathbb{T}^3)$ is an $\mathcal{F}_0$ measurable random variable. Then, there exists a unique local strong pathwise solution $\mathbf{v}$ to equation (2) satisfying*

$$\sup_{t\in[0,\tau_1)} \|\mathbf{v}(t)\|^2_{\mathbb{H}^1} + \int_0^{\tau_1} \|\mathbf{v}(t)\|_2^2 dt < \infty, \mathbb{P} - a.s.\ \omega \in \Omega,$$

*where $\tau_1$ is a positive random variable. Moreover, the local strong pathwise solution $v$ to equation (2) is Lipschitz continuous with respect to the initial data $\mathbf{u}_0$ in $\mathbb{H}^1(\mathbb{T}^3)$.*

# Step 3: Local well-posedness to Equation (2) in $\mathbb{H}^2(\mathbb{T}^3)$: $\mathbf{u}_0 \in \mathbb{H}^1(\mathbb{T}^3)$, $\mathbf{v}_n(s) \to \mathbf{v}(s)$ in $\mathbb{H}^2(\mathbb{T}^3), s \geq t_0 > 0$.

**Lemma 4 (Local well-posedness of strong solutions to Equation (2) in $\mathbb{H}^2(\mathbb{T}^3)$)**

*Suppose $\mathbf{u}_0 \in \mathbb{H}^2(\mathbb{T}^3)$ is an $\mathcal{F}_0$ measurable random variable. Then, there exists a unique local strong pathwise solution $\mathbf{v}$ to equation (2) on $[0,1]$ satisfying*

$$\sup_{t\in[0,\tau_2)} \|\mathbf{v}(t)\|_{\mathbb{H}^2}^2 + \int_0^{\tau_2} \|\mathbf{v}(t)\|_3^2 dt < \infty, \mathbb{P} - a.e.\ \omega \in \Omega.$$

*where the positive random variable $\tau_2$ is the local existence time for $\mathbf{v}$. Moreover, the local strong pathwise solution $\mathbf{v}$ to equation (2) is Lipschitz continuous with respect to the initial data in $\mathbb{H}^2(\mathbb{T}^3)$.*

# Sketch of proof

**Proof.** For $t \in (0, \tau_n)$, taking inner product of (3) in $L^2([0,t] \times \mathbb{T}^3)$ with $\Lambda^3 v_n$ yields

$$\frac{1}{2}\partial_t \|\mathbf{v}_n\|_2^2 + \|\mathbf{v}_n\|_3^2 = -\alpha^{-1}\langle(\mathbf{v}_n \cdot \nabla \mathbf{v}_n), \Lambda^4 \mathbf{v}_n\rangle$$

which implies

$$\begin{aligned} \|\mathbf{v}_n(t)\|_2^2 + 2\int_0^t \|\mathbf{v}_n(s)\|_3^2 ds \quad &\leq \quad \|\mathbf{u}_0\|_{\frac{3}{2}}^2 + \varepsilon \int_0^t \|\mathbf{v}_n(s)\|_3^2 ds \\ &\quad + c \sup_{s\in[0,t]} \alpha^{-2}(s) \sum_{i=1}^{3} \int_0^t \int_{\mathbb{T}^3} |\partial_{x_i}(\mathbf{v}_n \cdot \nabla \mathbf{v}_n)|^2 dx ds. \end{aligned}$$

Define $A := 1 + |u_0|_1^2$. By Poincaré inequality and standard argument, we have

$$\|v_n(t)\|_2^2 + \int_0^t \|v_n(s)\|_3^2 ds \leq c\|u_0\|_2^2 + c \sup_{s\in[0,1]} \alpha^{-2}(s) \int_0^t (A + \|v_n(s)\|_2^2)^3 ds.$$

# Sketch of proof

Again, by the comparison theorem

$$\|\mathbf{v}_n(t)\|_2^2 \leq \frac{A + \|\mathbf{u}_0\|_2^2}{\left[1 - 2c \sup_{s\in[0,1]} \alpha^{-2}(s)t(A + \|\mathbf{u}_0\|_2^2)^2\right]^{1/2}} - A. \quad (4)$$

Hence the estimates (4) rules out a blowup of $\mathbf{v}_n$ in $\mathbb{H}^2$ before the time $\tau_2^* = \frac{1}{2c \sup_{s\in[0,1]} \alpha^{-2}(s)(A+\|\mathbf{u}_0\|_2^2)^2}$. It follows that there exists $\tau_2 > 0$, we can for example take $\tau_2 = \tau_2^*/2$, such that $\tau_n \geq \tau_2$ for all $n$. From (4), we have uniform bounds for $\mathbf{v}_n$ in $L^\infty([0,\tau_2];\mathbb{H}^2(\mathbb{T}^3))$ and in $L^2([0,\tau_2];\mathbb{H}^3(\mathbb{T}^3))$. It is straightforward to show that $\partial_t\mathbf{v}_n$ is uniformly bounded in $L^2([0,\tau_2];\mathbb{L}^2(\mathbb{T}^3))$. One can obtain a subsequence of $\mathbf{v}_n$, which converges to $\mathbf{v}$ in $L^2([0,\tau_2];\mathbb{H}^2(\mathbb{T}^3))$ with $\mathbf{v} \in C([0,\tau_2];\mathbb{H}^2(\mathbb{T}^3))$. By a standard argument one knows $\mathbf{v}$ is a local strong solution to (2). The uniqueness of $\mathbf{v}$ is routine.

# Step 4: Global well-posedness of strong solutions

**Theorem 2.1 (Global well-posedness of strong solutions to Equation(2))**

*Suppose $\mathbf{u}_0 \in \mathbb{H}^1(\mathbb{T}^3)$ is an $\mathcal{F}_0$ measurable random variable. Then, for any $T > 0$, there exists a unique global strong pathwise solution $\mathbf{v}$ to (2).*

**Proof.**
Letting $n$ tend to infinite in step 1 via the convergence in step 2 and step 3 yields

$$\|\mathbf{v}(t)\|_1^2 \leq c\|\mathbf{v}(t_0)\|_1^2 \exp\left(c\|\mathbf{v}(t_0)\|_{\mathbb{H}^2}^2 \int_0^t \alpha^{-2}(r)dr\right),$$

where $t \in [0, \tau(\mathbf{u}_0, \omega))$. The global existence follows. The uniqueness is easy to be checked.

# Maximum principle to Equation (2)

## Proposition 5 (Maximum principle to equation (2))

*For any $\mathcal{F}_0$-adapted initial value $\mathbf{u}_0 \in \mathbb{H}^2(\mathbb{T}^3)$, let $(\mathbf{v}, \xi)$ be the maximum strong solution. Then for any $t \in (0, \xi)$, the solution $\mathbf{v}$ to (2) satisfies*

$$\sup_{s\in[0,t]} |\mathbf{v}(s)|_\infty \le |\mathbf{v}(0)|_\infty = |\mathbf{u}(0)|_\infty, \mathbb{P} - a.s. \omega \in \Omega.$$

**Sketch of proof:** By step 2, there exists a subsequence of solutions $\mathbf{v}_n$ such that

$$\mathbf{v}_n(s) \to \mathbf{v}(s) \text{ in } L^2([0,t]; \mathbb{H}^2(\mathbb{T}^3)).$$

Then we can choose a subsequence of $\mathbf{v}_n$ still denoted by $\mathbf{v}_n$ satisfying

$$\mathbf{v}_n(s) \to \mathbf{v}(s) \text{ in } \mathbb{L}^\infty(\mathbb{T}^3) \text{ for almost every } s \in [0,t].$$

Let $\varphi \in \mathbb{L}^1(\mathbb{T}^3)$ with $|\varphi|_1 \le 1$, we have

$$\langle \mathbf{v}(s), \varphi \rangle = \lim_{n\to\infty} \langle \mathbf{v}_n(s), \varphi \rangle \le \lim_{n\to\infty} |\mathbf{v}_n(s)|_\infty \le \lim_{n\to\infty} |\mathbf{v}_n(0)|_\infty \le |\mathbf{v}(0)|_\infty,$$

which implies

$$\sup_{s\in[0,t]} |\mathbf{v}(t)|_\infty \le |\mathbf{v}(0)|_\infty = |\mathbf{u}(0)|_\infty.$$

# Smooth solutions to Equation (2)

As an application of Proposition 5, we can further obtain the existence of smooth solution.

## Corollary 6 (Smooth solutions to Equation (2))

*Suppose* $\mathbf{u}_0 \in \mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)$ *is an* $\mathcal{F}_0$ *measurable random variable. Then, for any* $T > 0$, *there exists a unique global strong pathwise solution* $\mathbf{v}$ *to (2) satisfying*
$\mathbf{v} \in C([t_0, T]; \mathbb{H}^m(\mathbb{T}^3)) \cap L^2[t_0, T]; \mathbb{H}^{m+1}(\mathbb{T}^3)), \forall t_0 > 0, \forall m \geq 1.$

# Global well-posedness of weak solution to Equation (2)

## Global well-posedness of weak solution to Equation (2)

Suppose $\mathbf{u}_0 \in \mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)$ is an $\mathcal{F}_0$ measurable random variable. Then, there exists a unique global weak pathwise solution $\mathbf{v}$ to (2).

Ideal of the proof:

(i) Introducing an Ornstein-Uhlenbeck process to translate the problem into a new Cauchy problem with zero initial data, we get the local existence of weak solution.

(ii) Applying the global existence of the strong solution to the local solution, we establish the global existence of weak solution.

# Long-time behaviour of 3D Burgers equation

Due to its ubiquity, the Burgers equation is significant in the mathematical modelling of the large scale structure of the universe with complexity. Thus, it is natural and also very important to study long-time behaviour of the Burgers equation. However,

*No long-time behaviour of the deterministic equation can be derived.*

In fact, let $\mathbf{u}$ be the unique strong solution to the following deterministic 3D Burgers equation (perturbed by a linear damping term $b\mathbf{u}(t,x)$):

$$\partial_t \mathbf{u}(t,\mathbf{x}) - \Delta \mathbf{u}(t,\mathbf{x}) + (\mathbf{u}\cdot\nabla)\mathbf{u}(t,\mathbf{x}) = b\mathbf{u}(t,\mathbf{x}),$$
$$\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \in \mathbb{H}^2(\mathbb{T}^3), \quad \text{on } [0,T]\times\mathbb{T}^3.$$

If $b=0$, performing energy estimates in $\mathbb{L}^2(\mathbb{T}^3)$ space and applying the maximum principle, one gets

$$\partial_t |\mathbf{u}|_2^2 + \|\mathbf{u}\|_1^2 = \int_{\mathbb{T}^3} (\mathbf{u}\cdot\nabla)\mathbf{u}(t,\mathbf{x})\cdot\mathbf{u}(t,\mathbf{x})dx \le |\mathbf{u}_0|_\infty |\mathbf{u}|_2 \|\mathbf{u}\|_1.$$

By the Gronwall inequality,

$$|\mathbf{u}(t)|_2^2 \le |\mathbf{u}_0|_2^2 e^{|\mathbf{u}_0|_\infty^2 t}.$$

# Long-time behaviour of 3D Burgers equation

If $b \neq 0$, without Doss-Sussman transformation

$$\begin{aligned} &\partial_t \mathbf{u}(t,\mathbf{x}) - \Delta \mathbf{u}(t,\mathbf{x}) + (\mathbf{u}\cdot\nabla)\mathbf{u}(t,\mathbf{x}) = b\mathbf{u}(t,\mathbf{x}), \\ &\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \in \mathbb{H}^2(\mathbb{T}^3), \quad \text{on } [0,T]\times\mathbb{T}^3. \end{aligned}$$

Similar to the above argument, by maximum principle, we get

$$\partial_t |\mathbf{u}(t)|_2^2 + \|\mathbf{u}(t)\|_1^2 \quad \leq \quad (2b + 8|\mathbf{u}(t_0)|_\infty^2)|\mathbf{u}(t)|_2^2.$$

which implies

$$|\mathbf{u}(t)|_2^2 \leq |\mathbf{u}(0)|_2^2 \exp(2b + 8|\mathbf{u}(t_0)|_\infty^2)t.$$

If $2b + 8|\mathbf{u}(t_0)|_\infty^2 < 0$, then $|\mathbf{u}(t)|_2^2 \to 0$, as $t \to \infty$.

# Long-time behaviour of 3D Burgers equation with Doss-Sussman transformation

Next, applying the Doss-Sussman transformation to the 3D damped Burgers equation

$$\alpha(t) = \exp(-bt), \quad \mathbf{u}(t,x) = \alpha^{-1}(t)\mathbf{v}(t,\mathbf{x}), \; on \; [0,T] \times \mathbb{T}^3.$$

we get

$$\partial_t \mathbf{v} - \Delta \mathbf{v} + \alpha^{-1}\mathbf{v} \cdot \nabla \mathbf{v} = 0, \quad \mathbf{v}(0,\mathbf{x}) = u_0(\mathbf{x}), \; \mathbf{x} \in \mathbb{T}^3.$$

1. Applying maximum principle yields,

$$\|\mathbf{v}(t)\|_1^2 \quad \leq \quad c(\|\mathbf{v}(t_0)\|_{\mathbb{H}^2}^2, t_0) \exp\left(\int_0^t \exp(bs)ds\right),$$

2. if $b < 0$, then clearly

$$\sup_{t\in[0,\infty)} \|\mathbf{v}(t)\|_1^2 < \infty,$$

$$\|\mathbf{u}(t)\|_1^2 \quad \leq \quad c(\|\mathbf{u}(t_0)\|_{\mathbb{H}^2}^2, t_0)\underbrace{\exp\left(\int_0^t \exp(bs)ds\right)}_{\text{bounded!}} \exp(2bt) \to 0, \; as \; t \to \infty.$$

# Long-time behaviour of Itô type Equation (5)

For the 3D stochastic Burgers equation (with damping in Itô differential formulation)

$$\partial_t \mathbf{u}(t,\mathbf{x}) - \Delta \mathbf{u}(t,\mathbf{x}) + (\mathbf{u}\cdot\nabla)\mathbf{u}(t,\mathbf{x}) = b\mathbf{u}(t,\mathbf{x})dB(t),$$
$$\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \in \mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3), \quad \text{on } [0,T]\times\mathbb{T}^3. \tag{5}$$

Let us introduce the Doss-Sussman transformation

$$\alpha(t) := \exp(-bB(t) + \frac{b^2 t}{2}), \quad \mathbf{u}(t,x) = \alpha^{-1}(t)\mathbf{v}(t,\mathbf{x}), \ on\ [0,T]\times\mathbb{T}^3.$$

Then we get

$$d\mathbf{v}(t,\mathbf{x}) = \Delta\mathbf{v}(t,\mathbf{x})dt - \alpha^{-1}[(\mathbf{v}\cdot\nabla\mathbf{v})(t,\mathbf{x})]dt, \quad (t,\mathbf{x}) \in [0,\infty)\times\mathbb{T}^3, \tag{6}$$
$$\mathbf{v}(0,\mathbf{x}) = \mathbf{u}_0, \ \mathbf{x} = (x_1,x_2,x_3) \in \mathbb{T}^3.$$

Notice that $\lim_{t\to\infty}\alpha^{-1}(t) = 0$, which weakens the growth of the energy from the advection term.

# Long-time behaviour of Itô type Equation (5)

Similar to the above, we can get

## Proposition 7 (the maximum principle)

*Let* $\mathbf{v}$ *be the strong solution of the Burgers equation (6) on* $[0, T]$ *with initial data* $\mathbf{u}_0 \in \mathbb{H}^2(\mathbb{T}^3)$*, then we have* $\sup_{t\in[0,T]} |\mathbf{v}(t)|_\infty \leq |\mathbf{u}_0|_\infty$.

Taking advantage of the maximum principle of $\mathbf{v}$, we can show the following

## Theorem 8 (long-time behaviour for v and u)

*For* $t_0 > 0$*, and an* $\mathcal{F}_0$ *adapted initial value* $\mathbf{u}_0 \in \mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)$*,*

1. *the unique weak solution* $\mathbf{v}(t)$ *on* $t \in [t_0, \infty)$ *for arbitrary* $t_0 > 0$ *satisfies*

$$\|\mathbf{v}(t)\|_1^2 \quad \leq \quad c(\|\mathbf{v}(t_0)\|_{\mathbb{H}^2}^2, t_0, \omega),$$

2. *Consequently, the unique weak solution* $\mathbf{u}(t)$ *on* $t \in [t_0, \infty)$ *for arbitrary* $t_0 > 0$ *satisfies*

$$\|\mathbf{u}(t)\|_1^2 \quad \leq \quad c(\|\mathbf{u}(t_0)\|_{\mathbb{H}^2}^2, t_0, \omega) \exp\left(2bB(t) - b^2 t\right), \quad \mathbb{P} - a.s.\omega \in \Omega.$$

# Sketch of proof

Taking inner product of (6) in $\mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3)$ and integrating over $[t_0, t]$ for $t_0 \in (0, t)$ yields,

$$\|\mathbf{v}(t)\|_1^2 + 2\int_{t_0}^t \|\mathbf{v}(s)\|_2^2 ds \leq \|\mathbf{v}(t_0)\|_1^2 + 2\int_{t_0}^t \alpha^{-1}(s)|\mathbf{v}(s)|_\infty \|\mathbf{v}(s)\|_1 \|\mathbf{v}(s)\|_2 ds.$$

By the Hölder inequality, the Poincaré inequality, and the maximum principle, we get

$$\|\mathbf{v}(t)\|_1^2 + (2\lambda_1 - \varepsilon\lambda_1)\int_{t_0}^t \|\mathbf{v}(s)\|_1^2 ds \leq \|\mathbf{v}(t_0)\|_1^2 + c\|\mathbf{v}(t_0)\|_{\mathbb{H}^2}^2 \int_{t_0}^t \exp(2bB(s) - b^2 s)\|\mathbf{v}(s)\|_1^2 ds.$$

which implies

$$\|\mathbf{u}(t)\|_1^2 \quad \leq \quad c(\|\mathbf{u}(t_0)\|_{\mathbb{H}^2}^2, t_0, \omega)\exp\left(2bB(t) - b^2 t\right) \to 0, \ as \ t \to \infty. \qquad \square$$

Comparing to the 3D deterministic Burgers equation with damping $b\mathbf{u}, b < 0$,

$$\|\mathbf{u}(t)\|_1^2 \quad \leq \quad c(\|\mathbf{u}(t_0)\|_{\mathbb{H}^2}^2, t_0)\underbrace{\exp\left(\int_0^t \exp(bs)ds\right)}_{\text{bounded!}}\exp(2bt) \to 0, \ as \ t \to \infty,$$

a natural question is how about the damping term induced by a Stratonovich type noise?

# Long-time behaviour of Stratonovich type equation

We consider the 3D Burgers equation with noise in form of Stratonovich integral

$$\partial_t \mathbf{u}(t,\mathbf{x}) - \Delta \mathbf{u}(t,\mathbf{x}) + (\mathbf{u}\cdot\nabla)\mathbf{u}(t,\mathbf{x}) = b\mathbf{u}(t,\mathbf{x}) \circ dB(t),$$
$$\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \in \mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3), \quad \text{on } [0,T]\times\mathbb{T}^3.$$

Introducing the Doss-Sussman transformation

$$\alpha(t) := \exp\Big(-bB(t) + \frac{b^2 t}{2}\Big), \;\; \mathbf{u}(t,x) = \alpha^{-1}(t)\mathbf{v}(t,\mathbf{x}), \; on \; [0,T]\times\mathbb{T}^3,$$

we then get

$$d\mathbf{v}(t,\mathbf{x}) = \Delta\mathbf{v}(t,\mathbf{x})dt - \alpha^{-1}[(\mathbf{v}\cdot\nabla\mathbf{v})(t,\mathbf{x})]dt + \frac{1}{2}b^2\mathbf{v}dt, \quad (t,\mathbf{x}) \in [0,\infty)\times\mathbb{T}^3,$$
$$\mathbf{v}(0,\mathbf{x}) = \mathbf{u}_0, \;\; \mathbf{x} = (x_1,x_2,x_3) \in \mathbb{T}^3.$$

# Long-time behaviour of Stratonovich type equation

## the maximum principle for $\mathbf{v}$

Suppose $(\mathbf{v}, \xi)$ be a maximum strong solution of the Burgers equation above with initial data $\mathbf{u}_0 \in \mathbb{H}^2(\mathbb{T}^3)$ , then for arbitrary $s \in [0, \xi)$ we have

$$|\mathbf{v}(s)|_\infty \quad \leq \quad |\mathbf{u}_0|_\infty \exp\left(\frac{b^2 s}{2}\right).$$

**Sketch of proof:** Let $\beta > 0$ and set $f(s) := f(s,x) := e^{-\beta s - \frac{b^2 s}{2}} \mathbf{v}(s,x)$ for all $s \in [0, t]$ and $x \in \mathbb{T}^3$. Then, we have

$$\partial_s |f(s)|^2 + 2\beta |f(s)|^2 + e^{(\beta + \frac{b^2}{2})s} \alpha^{-1}(s) f(s) \cdot \nabla |f(s)|^2 - \Delta |f(s)|^2 + 2|\nabla f|^2 = 0.$$

Similar to the analysis before we get

$$|f(s)|_\infty \leq |f(0)|_\infty,$$

which implies

$$|\mathbf{v}(s)|_\infty \leq e^{(\beta + \frac{b^2}{2})s} |\mathbf{v}(0)|_\infty, \text{ for } s \in (0, t].$$

Let $\beta$ tends to $0$, we get the desired result.

# Estimates of Stratonovich type equation

Taking advantage of the maximum principle of $\mathbf{v}$ we prove that

## long-time behaviour for u

For $t_0 > 0$, and $\mathcal{F}_0$ adapted initial value $\mathbf{u}_0 \in \mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)$,

1. the unique weak solution $\mathbf{v}(t)$ on $t \in [t_0, \infty)$ for arbitrary $t_0 > 0$ satisfies

$$\begin{aligned} \|\mathbf{v}(t)\|_1^2 \quad \leq \quad & c(\|\mathbf{v}(t_0)\|_1^2, t_0, \omega) \exp\left( \|\mathbf{v}(t_0)\|_{\mathbb{H}^2}^2 \int_{t_0}^t \exp(2bB(s)) ds \right) \\ & \times \exp(b^2 - 2(1-\varepsilon)\lambda_1)(t - t_0). \end{aligned}$$

2. Consequently, the unique weak solution $\mathbf{u}(t)$ on $t \in [t_0, \infty)$ for arbitrary $t_0 > 0$ satisfies

$$\begin{aligned} \|\mathbf{u}(t)\|_1^2 \quad \leq \quad & c(\|\mathbf{u}(t_0)\|_1^2, t_0) \exp\left( \|\mathbf{v}(t_0)\|_{\mathbb{H}^2}^2 \int_{t_0}^t \Big[ \exp(2bB(s)) \Big] ds \right) \\ & \times \underbrace{\exp\left(2bB(t) - 2(1-\varepsilon)\lambda_1 t\right)}_{\to 0}. \end{aligned}$$

# Sketch of proof

**Sketch of proof:** Following the argument above, we get

$$\|\mathbf{v}(t)\|_1^2 + (2\lambda_1 - \varepsilon\lambda_1)\int_{t_0}^{t} \|\mathbf{v}(s)\|_1^2 ds \le \|\mathbf{v}(t_0)\|_1^2$$
$$+c\int_{t_0}^{t} \exp(2bB(s) - b^2 s)\exp(b^2 s)\|\mathbf{v}(s)\|_1^2\|\mathbf{v}(t_0)\|_{\mathbb{H}^2}^2 ds + b^2\|\mathbf{v}(t)\|_1^2.$$

By Gronwall inequality,

$$\|\mathbf{v}(t)\|_1^2 \le c(\|\mathbf{v}(t_0)\|_1^2)\exp\left(\|\mathbf{v}(t_0)\|_{\mathbb{H}^2}^2\int_{t_0}^{t}\exp(2bB(s))ds\right)$$
$$\times \exp(b^2 - 2(1-\varepsilon)\lambda_1)(t - t_0).$$

Subsequently,

$$\|\mathbf{u}(t)\|_1^2 \le c(\|\mathbf{v}(t_0)\|_1^2, t_0)\exp\left(\|\mathbf{v}(t_0)\|_{\mathbb{H}^2}^2\int_{t_0}^{t}\exp(2bB(s))ds\right)\exp\left(2bB(t) - 2(1-\varepsilon)\lambda_1 t\right) \to 0.$$

# Feller property for the solution $\mathbf{u}$ of Itô type Equation (5)

For $k \geq 1$, define

$$\tau_k(\mathbf{u}_0) = \inf_{t \geq 0}\{t : \int_0^t \|\mathbf{u}(s,\mathbf{u}_0)\|^2_{\mathbb{H}^{\frac{3}{2}}}(1 + \|\mathbf{u}(s,\mathbf{u}_0)\|^2_{\frac{1}{2}})ds \geq k\},$$

Since for $\mathbf{u}_0 \in \mathbb{H}^{\frac{1}{2}}$, $t\|\mathbf{u}(t)\|_1$ is continuous with respect to $t$, define

$$\sigma_j(\mathbf{u}_0) = \inf_{t \geq 0}\{t : t\|\mathbf{u}(t,\mathbf{u}_0)\|^2_1 \geq j\}.$$

Furthermore, define

$$\tau_k(\mathbf{u}_0, \tilde{\mathbf{u}}_0) = \tau_k(\mathbf{u}_0) \wedge \tau_k(\tilde{\mathbf{u}}_0), \quad \sigma_j(\mathbf{u}_0, \tilde{\mathbf{u}}_0) = \sigma_j(\mathbf{u}_0) \wedge \sigma_j(\tilde{\mathbf{u}}_0).$$

By delicate stopping time techniques and stochastic Gronwall inequality, we can obtain

## Lemma 9 (Lipschitz continuity)

*Let $t > 0$. Assume $\mathbf{u}_1$ and $\mathbf{u}_2$ are the solutions of (5) with initial data $\mathbf{u}_0, \tilde{\mathbf{u}}_0 \in \mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)$ respectively, which satisfy $\mathbb{E}\|\mathbf{u}_0\|^2_{\frac{1}{2}} < \infty$ and $\mathbb{E}\|\tilde{\mathbf{u}}_0\|^2_{\frac{1}{2}} < \infty$. Then we have*

$$\mathbb{E} \sup_{s \in [0, t \wedge \tau_k(\mathbf{u}_0, \tilde{\mathbf{u}}_0)]} \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|^2_{\mathbb{H}^{\frac{1}{2}}} \leq c(b,t,k)\mathbb{E}\|\mathbf{u}_0 - \tilde{\mathbf{u}}_0\|^2_{\mathbb{H}^{\frac{1}{2}}}.$$

# Feller property for Equation (5)

## Proposition 10 (Feller property for $\mathbf{u}$)

*The Markov semigroup $P_t$ associated to the 3D stochastic Burgers equation with deterministic initial data $\mathbf{u}_0 \in \mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)$ is Feller on $\mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)$, that is $P_t$ maps $C_b\left(\mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)\right)$ into itself.*

**Sketch of proof.** Fix $t > 0$, $\phi \in C_b(\mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3))$, $\mathbf{u}_0 \in \mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)$ , $\tilde{\mathbf{u}}_0 \in B_{\mathbb{H}^{\frac{1}{2}}}(1, \mathbf{u}_0)$ and $k, j \geq 1$,

$$
\begin{aligned}
& |\mathbb{E}\left(\phi(\mathbf{u}(t,\mathbf{u}_0)) - \phi(\mathbf{u}(t,\tilde{\mathbf{u}}_0))\right)| \\
\leq\ & |\mathbb{E}\left(\phi(\mathbf{u}(t,\mathbf{u}_0)) - \phi(\mathbf{u}(t,\tilde{\mathbf{u}}_0))\right) I_{\sigma_j(\mathbf{u}_0,\tilde{\mathbf{u}}_0)>t} I_{\tau_k(\mathbf{u}_0,\tilde{\mathbf{u}}_0)\geq t}| + 2|\phi|_\infty \mathbb{P}\{\tau_k(\mathbf{u}_0,\tilde{\mathbf{u}}_0) < t\} \\
& +2|\phi|_\infty \mathbb{P}\{\sigma_j(\mathbf{u}_0,\tilde{\mathbf{u}}_0) \leq t\} =: I_1 + I_2 + I_3.
\end{aligned}
$$

To estimate $I_1$, we introduce an element $\tilde{\phi} \in Lip\left(\mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)\right)$ to approximate the given $\phi \in C_b\left(\mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)\right)$. Then note that on the set $\left\{\sigma_j(\mathbf{u}_0,\tilde{\mathbf{u}}_0) > t\right\}$, one have $\mathbf{u}(t,\mathbf{u}_0), \mathbf{u}(t,\tilde{\mathbf{u}}_0) \in B_{\mathbb{H}^1}\left(\frac{j}{t}, \mathbf{0}\right)$ . Hence, for any $j, k > 1$, we obtain

$$
\begin{aligned}
I_1 &\leq 2 \sup_{u \in B_{\mathbb{H}^1}(\frac{j}{t},\mathbf{0})} |\phi(\mathbf{u}) - \tilde{\phi}(\mathbf{u})| + \left|\mathbb{E}\left(\tilde{\phi}(\mathbf{u}(t,\mathbf{u}_0)) - \tilde{\phi}(\mathbf{u}(t,\tilde{\mathbf{u}}_0))\right) I_{\tau_k(\mathbf{u}_0,\tilde{\mathbf{u}}_0)\geq t}\right| \\
&\leq 2 \sup_{\mathbf{u} \in B_{\mathbb{H}^1}(\frac{j}{t},\mathbf{0})} |\phi(\mathbf{u}) - \tilde{\phi}(\mathbf{u})| + C(L,t,b,k)\|\mathbf{u}_0 - \tilde{\mathbf{u}}_0\|_{\mathbb{H}^{\frac{1}{2}}} \to 0.
\end{aligned}
$$

# Sketch of proof

Note that $\mathbf{u} \in C([0,T]; \mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)) \cap \mathbb{L}^2([0,T]; \mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3))$ and $t\mathbf{u}(t) \in C([0,T]; \mathbb{H}^1(\mathbb{T}^3))$, we immediately get that

$$\tau_k(\mathbf{u}_0, \tilde{\mathbf{u}}_0) \to \infty, \text{ as } k \to \infty \text{ and } \sigma_j(\mathbf{u}_0, \tilde{\mathbf{u}}_0) \to \infty, \text{ as } j \to \infty$$

which in turn implies

$$I_2 + I_3 \to 0, \text{ as } k, j \to \infty.$$

## Theorem 2.2 (Ergodicity for u)

*Given $\mathcal{F}_0$ adapted initial data $\mathbf{u}_0 \in \mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)$ with $\mathbb{E}\|\mathbf{u}_0\|^2_{\mathbb{H}^{\frac{1}{2}}} < \infty$, then $\delta_0$ is the unique invariant measure to 3D stochastic Burgers equation (5).*

# Outline

# Global well-posedness with diffusion coefficient $b$ being a spatial function

We consider 3D Burgers equation (1) with $b(x)$ being a given smooth function of the space variable.

$$
\begin{aligned}
&d\mathbf{u}(t,x) = \Delta\mathbf{u}(t,x)dt - ((\mathbf{u}\cdot\nabla)\mathbf{u}(t,x))dt + \mathbf{u}(t,x)\circ b(x)dB(t), \quad \text{on } [0,T]\times\mathbb{T}^3, \\
&\mathbf{u}(0,x) = \mathbf{u}_0(x), \quad x = (x_1,x_2,x_3)\in\mathbb{T}^3,
\end{aligned} \tag{7}
$$

where $b(x) : \mathbb{T}^3 \ni x \to \mathbb{R},$ is a given smooth function.

# Reformulation of (7)

Let

$$\lambda = \sup_{(t,x)\in[0,T]\times\mathbb{T}^3} \left[ \left( |\sum_{i=1}^{3} \partial_{x_i} b(x)B(t)| \right)^2 + |\Delta b(x)B(t)| \right) \Big].$$

$$\hat{\mathbf{v}}(t,x) \quad = \quad \mathbf{u}(t,x) \exp\Big(-b(x)B(t)\Big)\exp(-\lambda t) =: \mathbf{u}(t,x)\alpha(t,x)\exp(-\lambda t).$$

(7) is equivalent to the following

$$\begin{aligned}
&\partial_t \hat{\mathbf{v}}(t,x) - \Delta\hat{\mathbf{v}}(t,x) - 2\sum_{i=1}^{3}\Big(\partial_{x_i} b(x)B(t)\Big)\partial_{x_i}\hat{\mathbf{v}}(t,x) \\
&+\alpha^{-1}(t,x)\exp(\lambda t)\sum_{i=1}^{3}\hat{\mathbf{v}}_i(t,x)\partial_{x_i}\hat{\mathbf{v}}(t,x) \\
&+\Big(\lambda - \Big(\sum_{i=1}^{3}\partial_{x_i} b(x)B(t)\Big)^2 - \Big(\Delta b(x)B(t)\Big)\Big)\hat{\mathbf{v}}(t,x) \\
&+\alpha^{-1}(t,x)\exp(\lambda t)\Big(\sum_{i=1}^{3}\hat{v}_i(t,x)\partial_{x_i} b(x)B(t)\Big)\hat{\mathbf{v}}(t,x) = 0, \\
&\hat{\mathbf{v}}(0,x) = \mathbf{u}_0(x), \quad \text{on } [0,T]\times\mathbb{T}^3,
\end{aligned} \tag{8}$$

where $\hat{\mathbf{v}}(t,x) = (\hat{v}_1(t,x), \hat{v}_2(t,x), \hat{v}_3(t,x)) \in \mathbb{R}^3, x \in \mathbb{T}^3, \ t \in [0,T]$.

# Solving (8)

The global well-posedness of 3D deterministic Burgers equation rests upon the maximum principle according to the frame of deterministic 3D Burgers equation (see page 11 of (J.C.Robinson etc. "The three dimensional Navier-Stokes equation, 2016")). By virtue of the maximum principle (see Theorem 4 page 353 (L.C. Evans, "PDEs, 2016")), the coefficient of $\hat{\mathbf{v}}$ should be nonnegative. That is to say, one needs that

$$B(t)\sum_{i=1}^{3}\hat{v}_i(t,x)\partial_{x_i}b(x)\geq 0,\ \text{on } [0,T]\times\mathbb{T}^3,$$

or equivalently,

$$B(t)\sum_{i=1}^{3}u_i(t,x)\partial_{x_i}b(x)\geq 0,\ \text{on } [0,T]\times\mathbb{T}^3. \tag{9}$$

where $\mathbf{u}(t,x)=(u_1(t,x),u_2(t,x),u_3(t,x))\in\mathbb{R}^3, x\in\mathbb{T}^3,\ t\in[0,T]$. Since Brownian motion $B(t)$ visits zero at any finite time interval with positive probability, (9) is only possible when

$$\hat{\mathbf{v}}(t,x)\cdot\nabla b(x)=0,\ \ on\ [0,T]\times\mathbb{T}^3,\mathbb{P}-a.s.. \tag{10}$$

# Solving (8)

For general $b(x)$, it may result in $\hat{\mathbf{v}}_n(t,x) \equiv 0,\ t \in [0,T], x \in \mathbb{T}^3$. Hence, to find a nontrivial global solution to (8), we need following assumption:

Two compoments of $\nabla b(x) = (\partial_{x_1} b(x), \partial_{x_2} b(x), \partial_{x_3} b(x))$ are linearly correlated.

Without loss of generality, we assume there exists some constant $\lambda_1$ and $\lambda_2 \in \mathbb{R}$, such that

$$\lambda_1 \partial_{x_1} b(x) + \lambda_2 \partial_{x_2} b(x) - \partial_{x_3} b(x) = 0, \ \textit{holds for all } x \in \mathbb{T}^3. \tag{11}$$

For arbitrary $a \in \mathbb{R}$, let $\eta = (a\lambda_1, a\lambda_2, -a) \in \mathbb{R}^3$. In the following we will find a solution $\tilde{\mathbf{v}}(t,x) = g(t,x)\eta$ to (8), where $g : [0,T] \times \mathbb{T}^3 \to \mathbb{R}$ is determined by the equation (12):

# Solving (8)

$$\partial_t g(t,x) - \Delta g(t,x) - 2\sum_{i=1}^{3}\Big(\partial_{x_i} b(x)B(t)\Big)\partial_{x_i} g(t,x) \tag{12}$$
$$+\alpha^{-1}(t,x)\exp(\lambda t)(g(t,x)\partial_{x_1} - ag(t,x)\partial_{x_2})g(t,x)$$
$$\Big(\lambda - \Big(\sum_{i=1}^{3}\partial_{x_i} b(x)B(t)\Big)^2 - \Delta b(x)B(t)\Big)g(t,x) = 0, \quad \text{on } [0,T]\times\mathbb{T}^3,$$
$$g(0,x) := g_0 \in H^1(\mathbb{T}^3), \quad x = (x_1,x_2,x_3) \in \mathbb{T}^3.$$

Similar to the argument of local well-posedness of (2), we have

## local well-posedness of (12)

Suppose $g_0 \in H^1(\mathbb{T}^3)$ is an $\mathcal{F}_0$ measurable random variable. Then, there exists a unique maximum strong solution $(g,\xi)$ to the equation (12).

As a consequent result, we have

## local well-posedness of (8)

Suppose $\tilde{v}(0,x) = (\lambda g_0, -a\lambda g_0, 0) \in \mathbb{H}^1(\mathbb{T}^3)$ is an $\mathcal{F}_0$ measurable random variable. Then, there exists a unique maximum strong solution $(\tilde{v},\xi)$ to the equation (8).

# Solving (8)

Similar to the argument of local well-posedness of (2), we have

## local well-posedness of (12)

Suppose $g_0 \in H^1(\mathbb{T}^3)$ is an $\mathcal{F}_0$ measurable random variable. Then, there exists a unique maximum strong solution $(g, \xi)$ to the equation (12).

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Suppose $\tilde{v}(0, x) = (\lambda g_0, -a\lambda g_0, 0) \in \mathbb{H}^1(\mathbb{T}^3)$ is an $\mathcal{F}_0$ measurable random variable. Then, there exists a unique maximum strong solution $(\tilde{v}, \xi)$ to the equation (8).

# Global well-posedness of (8)

## Maximum principle for (8)

Let $(\tilde{\mathbf{v}}, \xi)$ be a maximum strong solution to (8) with $\mathcal{F}_0$ measurable initial data $u_0 = (\lambda g_0, -a\lambda g_0, 0) \in \mathbb{H}^2(\mathbb{T}^3)$. Then the solution $\tilde{\mathbf{v}}$ to (8) with Condition (11) satisfies

$$\sup_{t\in[0,\xi)} |\tilde{\mathbf{v}}(t)|_\infty \leq |\mathbf{v}(0)|_\infty = |\mathbf{u}(0)|_\infty, \mathbb{P} - a.s.\omega \in \Omega.$$

With the local well-posedness and maximum principle for (8), we arrive at

## Global existence and uniqueness of strong solutions to (8)

Suppose $\mathbf{u}_0 = (\lambda g_0, -a\lambda g_0, 0) \in \mathbb{H}^1(\mathbb{T}^3)$ is an $\mathcal{F}_0$ measurable random variable. Then, for any $T > 0$, there exists a unique global strong solution $\tilde{\mathbf{v}}$ to (8) with Condition (11).

## Global existence and uniqueness of weak solution to (8)

Suppose $\mathbf{u}_0 = (\lambda g_0, -a\lambda g_0, 0) \in \mathbb{H}^{\frac{1}{2}}(\mathbb{T}^3)$ is an $\mathcal{F}_0$ measurable random variable. Then, there exists a unique global weak pathwise solution $\tilde{\mathbf{v}}$ to (8) with Condition (11).

Thank You For Your Attention!